数理解析研究所講究録
964巻 1996年 40-50

ON MACAULAYFICATION OF LOCAL RINGS —IN THE CASE OF $\dim \text{non-CM} \leq 2$

KAWASAKI, TAKESI

Tokyo Metropolitan University

ABSTRACT. Let X be a Noetherian scheme. A birational proper morphism $Y \to X$ is said to be a Macaulayfication of X if Y is a Cohen-Macaulay scheme. In 1978 Faltings constructed a Macaulayfication of X if the dimension of its non-Cohen-Macualay locus non-CM X is at most one. Recently the author constructed a Macaulayfication of X in the case of non-CM $X = 2$. In the present article, we give another proof of them, which still work in general case except for only one lemma.

1. INTRODUCTION

Let X be a Noetherian scheme. A *Macaulayfication* of X is a birational proper morphism $Y \to X$ such that Y is a Cohen-Macaulay scheme. If $X = \operatorname{Spec} A$ is an affine scheme, then by abuse notation the Macaulayfication $Y \to \operatorname{Spec} A$ is said to be the one of A. In 1978, Faltings [4] gave the notion of Macaulayfication and constructed a Macaulayfication of Noetherian local ring A if it possesses a dualizing complex and $\dim \text{non-CM}\, A \leq 1$. Here $\text{non-CM}\, A = \{\mathfrak{p} \in \operatorname{Spec} A \mid A_{\mathfrak{p}} \text{ is not Cohen-Macaulay}\}$ is the non-Cohen-Macaulay locus of A, which is closed subset of $\operatorname{Spec} A$ if A possesses a dualizing complex. In the present article, we will construct a Macaulayfication of a Noetherian local ring A in the case of $\dim \text{non-CM}\, A \leq 2$.

Theorem 1.1 ([9]). *Let A be a Noetherian local ring possessing a dualizing complex. If $\operatorname{Ass} A = \operatorname{Assh} A$ and $\dim \text{non-CM}\, A \leq 2$, then A has a Macaulayfication.*

Here $\operatorname{Ass} A$ denotes the set of associated prime ideals of A and $\operatorname{Assh} A = \{\mathfrak{p} \in \operatorname{Ass} A \mid \dim A/\mathfrak{p} = \dim A\}$.

The notion of Macaulayfication is an analogue of the resolution of singularities. In 1964, Hironaka [8] gave a resolution of singularities of an algebraic variety over a field of characteristic zero. However the general resolution problem is still open even a variety over a field of positive characteristic. On the other hand, Faltings'

1991 *Mathematics Subject Classification.* Primary: 14M05, 14B05; Secondary: 13H10, 13A30.
Key words and phrases. blowing-up, Cohen-Macaulay scheme, dualizing complex, resolution of singularities.

method to construct a Macaulayfication is independent of the characteristic of A. In particular, it still works if A is mixed characteristic. Of course, our method is also independent of the characteristic.

In the last section, we give an application of Macaulayfication. A dualizing complex is an important tool of Commutative Algebra and Algebraic Geometry, though we know what rings possesses it not well. It is well-known that a homomorphic image of a Gorenstein local ring possesses a dualizing complex. In 1979, Sharp asked whether its converse is true [14]. Aoyama and Goto [1] gave a partial answer to Sharp's question by using Faltings' Macaulayfication. They showed that Sharp's question is true for a rings with $\dim \text{non-CM} \leq 1$. Their argument still works in the case of $\dim \text{non-CM} = 2$. We will show the following theorem.

Theorem 1.2. *Let A be a Noetherian local ring possessing a dualizing complex. If $\dim \text{non-CM}\, A \leq 2$, then A is a homomorphic image of a Gorenstein local ring.*

Throughout this article, A denotes a Noetherian local ring with maximal ideal $\mathfrak{m}$. Assume that $d = \dim A > 0$.

2. A system of parameters

In this section, we state on the p-standard system of parameters, which was introduced by Cuong [2]. First we recall the definition of u.s.d-sequences.

Definition 2.1 ([7]). Let M be an A-module. A sequence $x_1, \ldots, x_u \in A$ is said to be a d-sequence on M if

$$(x_1, \ldots, x_{i-1})M : x_i x_j = (x_1, \ldots, x_{i-1})M : x_j \quad \text{for any } 1 \leq i \leq j \leq u.$$

A sequence $x_1, \ldots, x_u$ is said to be a *u.s.d-sequence* on M if $x_1^{n_1}, \ldots, x_u^{n_u}$ is a d-sequence on M for any integers $n_1, \ldots, n_u > 0$ and in any order.

The following definition and lemmas are useful to find a u.s.d-sequence, which were given by Schenzel [12, 13].

Definition 2.2. For any finitely generated A-module M, let $\mathfrak{a}_i(M)$ be the annihilator of $H^i_{\mathfrak{m}}(M)$ and $\mathfrak{a}(M) = \prod_{i \neq \dim M} \mathfrak{a}_i(M)$.

Lemma 2.3. *Let M be a finitely generated A-module. If A possesses a dualizing complex, then the following statements are true:*

1. *For all i, $\dim A/\mathfrak{a}_i(M) \leq i$. In particular, $\dim A/\mathfrak{a}(M) < \dim M$.*
2. *Let $\mathfrak{p}$ be a prime ideal of A such that $\dim A/\mathfrak{p} = i$. Then $\mathfrak{p} \in \operatorname{Ass} M$ if and only if $\mathfrak{p} \in \operatorname{Ass} A/\mathfrak{a}_i(M)$. In particular, $\operatorname{Ass} M = \operatorname{Assh} M$ if and only if $\dim A/\mathfrak{a}_i(M) < i$ for all $i < \dim M$.*
3. *If M is equidimensional, then $\text{non-CM}\, M = V(\mathfrak{a}(M))$.*

Lemma 2.4. *Let M be a finitely generated A-module and $x_1, \dots, x_u$ a system of parameters for M. Then*

$$(x_1, \dots, x_{i-1})M : x_i \subseteq (x_1, \dots, x_{i-1})M : \mathfrak{a}(M) \quad \textit{for all } 1 \leq i \leq n.$$

The following definition is slightly different from Cuong's one.

Definition 2.5. Let M be a finitely generated A-module and $x_1, \dots, x_u$ is a system of parameters for M. We say that $x_1, \dots, x_u$ is a *p-standard system of parameters* of type s if

$$\begin{cases} x_{s+1}, \dots, x_u \in \mathfrak{a}(M) \\ x_i \in \mathfrak{a}(M/(x_{i+1}, \dots, x_u)M) \quad \text{for } i \leq s. \end{cases}$$

If A possesses a dualizing complex and $s \leq \dim A/\mathfrak{a}(M)$, then we can take a p-standard system of parameters of type s for M by using (1) of Lemma 2.3.

The following is the main theorem of this section, which was given by Cuong in his unpublished work.

Theorem 2.6. *Let M be a finitely generated A-module, $x_1, \dots, x_u$ its p-standard system of parameters of type s and $t \leq u$ a positive integer. Then $x_t^{n_t}, \dots, x_u^{n_u}$ is a d-sequence on M for any integers $n_t, \dots, n_u > 0$.*

Proof. We have to prove that

$$(2.6.1) \qquad (x_t^{n_t}, \dots, x_{i-1}^{n_{i-1}})M : x_i^{n_i} x_j^{n_j} = (x_t^{n_t}, \dots, x_{i-1}^{n_{i-1}})M : x_j^{n_j}$$

for any $t \leq i \leq j \leq u$. If $j \geq s+1$, then the both side of (2.6.1) equal to $(x_t^{n_t}, \dots, x_{i-1}^{n_{i-1}})M : \mathfrak{a}(M)$.

Assume that $j \leq s$ and take an element a of the left hand side of (2.6.1). Then

$$\begin{aligned} a &\in (x_t^{n_t}, \dots, x_{i-1}^{n_{i-1}}, x_{j+1}, \dots, x_d)M : x_i^{n_i} x_j^{n_j} \\ &= (x_t^{n_t}, \dots, x_{i-1}^{n_{i-1}}, x_{j+1}, \dots, x_d)M : x_j^{n_j}. \end{aligned}$$

Thus we have

$$x_j^{n_j} a \in (x_t^{n_t}, \dots, x_{i-1}^{n_{i-1}}, x_{j+1}, \dots, x_d)M \cap (x_t^{n_t}, \dots, x_{i-1}^{n_{i-1}})M : x_i^{n_i}.$$

The following lemma assures us that the right hand side of this equation is equal to $(x_t^{n_t}, \dots, x_{i-1}^{n_{i-1}})M$. $\square$

Lemma 2.7. *In the same notation as Theorem 2.6,*

(2.7.1)
$$(x_t^{n_t}, \dots, x_{i-1}^{n_{i-1}}, x_{j+1}, \dots, x_u)M \cap (x_t^{n_t}, \dots, x_{i-1}^{n_{i-1}})M : x_i^{n_i} = (x_t^{n_t}, \dots, x_{i-1}^{n_{i-1}})M$$

for all $t \leq i \leq j \leq u$.

Proof. We work by descending induction on j. If $j = u$, then there is is nothing to prove. Assume that $j < u$ and let a be an element of the left hand side of (2.7.1). Then $a = b + x_{j+1}c$ with $b \in (x_t^{n_t}, \dots, x_{i-1}^{n_{i-1}}, x_{j+2}, \dots x_u)M$ and $c \in M$. By using Lemma 2.4, we have

$$\begin{aligned} c &\in (x_t^{n_t}, \dots, x_{i-1}^{n_{i-1}}, x_{j+2}, \dots x_u)M : x_i^{n_i} x_{j+1} \\ &= (x_t^{n_t}, \dots, x_{i-1}^{n_{i-1}}, x_{j+2}, \dots x_u)M : x_{j+1}. \end{aligned}$$

Hence

$$\begin{aligned} a &\in (x_t^{n_t}, \dots, x_{i-1}^{n_{i-1}})M : x_i^{n_i} \cap (x_t^{n_t}, \dots, x_{i-1}^{n_{i-1}}, x_{j+2}, \dots x_u)M \\ &= (x_t^{n_t}, \dots, x_{i-1}^{n_{i-1}})M \end{aligned}$$

by induction hypothesis. □

3. The Proof of Theorem 1.1

The main theorem of this section is the following

Theorem 3.1. *Assume that $d \geq 2$ and there is a subsystem of parameters $x_t, \dots, x_d$ for A satisfying the following two conditions for some integer $s \geq t-1$:*

(#) $x_t^{n_t}, \dots, x_s^{n_s}, x_{\sigma(s+1)}^{n_{s+1}}, \dots, x_{\sigma(d)}^{n_d}$ *is a d-sequence on A for any positive integers $n_t, \dots, n_d$ and for any permutation σ of $s+1, \dots, d$.*

(%) $x_t, \dots, x_i$ *is a d-sequence on $A/(x_{i+1}, \dots, x_d)$ for all $t \leq i \leq s+1$.*

We put $\mathfrak{q}_i = (x_i, \dots, x_d)$, $\mathfrak{b}_i = \mathfrak{q}_i \cdots \mathfrak{q}_{s+1}$ and $X_i = \operatorname{Proj} A[\mathfrak{b}_i T]$ for $t \leq i \leq s+1$, where T is an indeterminate.

If $s-1 \leq t \leq s+1$, then $\operatorname{depth} \mathcal{O}_{X_t,p} \geq d-t+1$ for all closed point $p \in X_t$.

Theorem 1.1 immediately comes from Theorem 3.1. In fact, if $d \leq 1$, then A itself is Cohen-Macaulay. If $d \geq 2$, then $s = \dim \text{non-CM}\, A \leq d-2$ by (2) and (3) of Lemma 2.3. Let $x_1, \dots, x_d$ be a p-standard system of parameters of type s for A. Theorem 2.6 says that $x_1, \dots, x_d$ satisfies (#) and (%). Hence X_1 is a Cohen-Macaulay scheme.

The rest of this section is devoted to the proof of Theorem 3.1. From now on, we use the notation of Theorem 3.1. Of course, $x_{t+1}, \dots, x_d$ satisfy (#) and (%) as a system of parameters for $A/x_t^l A$ for any positive integer $l \leq s+1$. Furthermore, they satisfy (#) and (%) as a system of parameters for A. For example, we get

$$\begin{aligned} (x_{t+1}, \dots, x_{i-1}) : x_i x_j &= \bigcap_l (x_t^l, x_{t+1}, \dots, x_{i-1}) : x_i x_j \\ &= \bigcap_l (x_t^l, x_{t+1}, \dots, x_{i-1}) : x_j \\ &= (x_{t+1}, \dots, x_{i-1}) : x_j \end{aligned}$$

by Krull's intersection theorem.

Lemma 3.2. *Let $y_0, \dots, y_u \in A$. If $y_1, \dots, y_u$ is a d-sequence on A/y_0A, then*

$$(3.2.1) \qquad (y_1, \dots, y_k)(y_1, \dots, y_u)^n : y_0 = (y_1, \dots, y_k)[(y_1, \dots, y_u)^n : y_0] + 0 : y_0$$

for all $n > 0$ and $1 \leq k \leq u$.

Proof. We work by induction on k. Let $k = 1$ and a an element of the left hand side of (3.2.1). Then $y_0 a = y_1 b$ with $b \in (y_1, \dots, y_u)^n$. By using Theorem 1.3 of [7], $b \in (y_0) : y_1 \cap (y_1, \dots, y_u)^n \subseteq (y_0)$. If we put $b = y_0 a'$, then $a' \in (y_1, \dots, y_u)^n : y_0$ and $a - y_1 a' \in 0 : y_0$. Thus a belongs to the right hand side of (3.2.1).

Assume that $k \geq 2$ and let a be an element of the left hand side of (3.2.1). We put $y_0 a = y_k b + b'$ with $b \in (y_1, \dots, y_u)^n$ and $b' \in (y_1, \dots, y_{k-1})(y_1, \dots, y_u)^n$. Then we have

$$\begin{aligned} c &\in (y_0, y_1, \dots, y_{k-1}) : y_k \cap [(y_0) + (y_1, \dots, y_u)^n] \\ &= (y_0) + (y_1, \dots, y_{k-1})(y_1, \dots, y_u)^{n-1} \end{aligned}$$

by using Theorem 1.3 of [7] again. Let

$$b = y_0 a' + c$$

with $c \in (y_1, \dots, y_{k-1})(y_1, \dots, y_u)^{n-1}$. Then $a' \in (y_1, \dots, y_u)^n : y_0$ and

$$\begin{aligned} a - y_k a' &\in (y_1, \dots, y_{k-1})(y_1, \dots, y_u)^n : y_0 \\ &= (y_1, \dots, y_{k-1})[(y_1, \dots, y_u)^n : y_0] + 0 : y_0 \end{aligned}$$

by induction hypothesis. The proof is completed. □

The following is a bottle neck of the general Macaulayfication problem.

Proposition 3.3. *If $i = s$ or $s + 1$, then*

$$\mathfrak{q}_{i-1}[\mathfrak{b}_i^n : x_{i-1}^l] \subseteq \mathfrak{b}_i^n \quad \text{for all } n > 0 \text{ and } l > 0.$$

Proof. Assume that $i = s + 1$. Then Lemma 3.2 says that

$$\begin{aligned} \mathfrak{q}_{s+1}^n : x_s^l &= \mathfrak{q}_{s+1}^{n-1}[\mathfrak{q}_{s+1} : x_s^l] + 0 : x_s^l \\ &= \mathfrak{q}_{s+1}^{n-1}[\mathfrak{q}_{s+1} : x_s] + 0 : x_s. \end{aligned}$$

Thus we have the assertion.

Next assume that $i = s$. We prove

$$(3.3.1) \qquad \mathfrak{b}_s^n : x_{s-1}^l = \mathfrak{b}_s^{n-1}\mathfrak{q}_{s+1}[\mathfrak{q}_s : x_{s-1}] + x_s^n \mathfrak{q}_{s+1}^{n-1}[\mathfrak{q}_{s+1} : x_{s-1}] + 0 : x_{s-1}$$

for all $n > 0$ and $l > 0$. Let a be an element of the left hand side of (3.3.1). Then by Lemma 3.2, we have

$$\begin{aligned} a &\in \mathfrak{q}_s^{2n} : x_{s-1}^l \\ &= \mathfrak{q}_s^{2n-1}[\mathfrak{q}_s : x_{s-1}] + 0 : x_{s-1} \\ &= \mathfrak{q}_s^{n-1}\mathfrak{q}_{s+1}^n[\mathfrak{q}_s : x_{s-1}] + 0 : x_{s-1} + x_s^n \mathfrak{q}_s^{n-1}[\mathfrak{q}_s : x_{s-1}]. \end{aligned}$$

Hence we may assume that $a = x_s^n a'$ with $a' \in \mathfrak{q}_s^{n-1}[\mathfrak{q}_s : x_{s-1}]$. Then

$$x_{s-1}^l x_s^n a' \in \mathfrak{q}_{s+1}^{2n} + \cdots + x_s^{n-1}\mathfrak{q}_{s+1}^{n+1} + x_s^n \mathfrak{q}_{s+1}^n.$$

We put $x_{s-1}^l x_s^n a' = b + x_s^n b'$ with $b \in \mathfrak{q}_{s+1}^{n+1}$ and $b' \in \mathfrak{q}_{s+1}^n$. Since

$$\begin{aligned} x_{s-1}^l a' - b' &\in \mathfrak{q}_{s+1}^{n+1} : x_s^n \cap \mathfrak{q}_s \\ &= \mathfrak{q}_{s+1}^n[\mathfrak{q}_{s+1} : x_s] + 0 : x_s \cap \mathfrak{q}_s \\ &\subset \mathfrak{q}_{s+1}^n. \end{aligned}$$

Therefore

$$a' \in \mathfrak{q}_{s+1}^n : x_{s-1}^l = \mathfrak{q}_{s+1}^{n-1}[\mathfrak{q}_{s+1} : x_{s-1}] + 0 : x_{s-1}$$

by Lemma 3.2. Thus (3.3.1) is proved and the assertion comes from it. $\square$

Next we consider affine charts of X_i. We put

$$\begin{aligned} \mathfrak{c}_i = {} & (x_{s+1}^{s-i+2}, \ldots, x_d^{s-i+2}) \\ & + (x_{\alpha_1}^{\alpha_1 - i+1} x_{\alpha_2}^{\alpha_2-\alpha_1} \cdots x_{\alpha_{k-1}}^{\alpha_{k-1}-\alpha_{k-2}} x_{\alpha_k}^{s-\alpha_{k-1}+1} \mid i \le \alpha_1 < \cdots < \alpha_{k-1} \le s < \alpha_k) \end{aligned}$$

for all $t \le i \le s+1$.

Lemma 3.4. *The ideal $\mathfrak{c}_i$ is a reduction of $\mathfrak{b}_i$, that is, $\mathfrak{b}_i^n = \mathfrak{c}_i\mathfrak{b}_i^{n-1}$ for a sufficiently large n.*

Proof. We work by descending induction on i. If $i = s+1$, then $\mathfrak{b}_{s+1} = \mathfrak{c}_{s+1} = \mathfrak{q}_{s+1}$. There is nothing to prove.

Assume that $i \le s$ and $\mathfrak{b}_j^n = \mathfrak{c}_j\mathfrak{b}_j^{n-1}$ for all $i < j \le s+1$ and for a sufficiently large n. Let k be an integer such that $0 \le k \le s-i$. Then, since $x_{i+k}^{k+1}\mathfrak{c}_{i+k+1} \subset \mathfrak{c}_i$, we have

$$\begin{aligned} \mathfrak{q}_{i+1}\cdots\mathfrak{q}_{i+k-1}^{k-1}\mathfrak{q}_{i+k}^{kn-\binom{k}{2}}\mathfrak{b}_{i+k}^n &= \mathfrak{q}_{i+1}\cdots\mathfrak{q}_{i+k-1}^{k-1}\mathfrak{q}_{i+k}^{(k+1)n-\binom{k}{2}}\mathfrak{b}_{i+k+1}^n \\ &= \mathfrak{q}_{i+1}\cdots\mathfrak{q}_{i+k}^{k}\mathfrak{q}_{i+k+1}^{(k+1)n-\binom{k+1}{2}}\mathfrak{b}_{i+k+1}^n \\ &\quad + \mathfrak{q}_{i+1}\cdots\mathfrak{q}_{i+k-1}^{k-1}\mathfrak{q}_{i+k}^{(k+1)(n-1)-\binom{k}{2}}[x_{i+k}^{k+1}\mathfrak{c}_{i+k+1}\mathfrak{b}_{i+k+1}^{n-1}] \\ &\subseteq \mathfrak{q}_{i+1}\cdots\mathfrak{q}_{i+k}^{k}\mathfrak{q}_{i+k+1}^{(k+1)n-\binom{k+1}{2}}\mathfrak{b}_{i+k+1}^n + \mathfrak{c}_i\mathfrak{b}_i^{n-1}. \end{aligned}$$

Hence

$$\begin{aligned}
\mathfrak{b}_i^n &= \mathfrak{q}_i^n \mathfrak{b}_{i+1}^n \\
&\subseteq \mathfrak{q}_{i+1}^n \mathfrak{b}_{i+1}^n + \mathfrak{c}_i \mathfrak{b}_i^{n-1} \\
&\subseteq \mathfrak{q}_{i+1} \mathfrak{q}_{i+2}^{2n-1} \mathfrak{b}_{i+2}^n + \mathfrak{c}_i \mathfrak{b}_i^{n-1} \\
&\cdots \\
&\subseteq \mathfrak{q}_{i+1} \mathfrak{q}_{i+2}^2 \cdots \mathfrak{q}_s^{s-i} \mathfrak{q}_{s+1}^{(s-i+2)n-\binom{s-i+1}{2}} + \mathfrak{c}_i \mathfrak{b}_i^{n-1} \\
&= \mathfrak{c}_i \mathfrak{b}_i^{n-1}
\end{aligned}$$

because $(x_{s+1}^{s-i+2}, \ldots, x_d^{s-i+2}) \subset \mathfrak{c}_i$ is a reduction of $\mathfrak{q}_{s+1}$. $\square$

Thus X_i is covered by spectrum of such rings as

$$A[\mathfrak{b}_i/x_\alpha^{s-i+2}] = A\left[\frac{x_i}{x_\alpha}, \ldots, \frac{x_d}{x_\alpha}\right]$$

with $s+1 \leq \alpha \leq d$ and

$$\begin{aligned}
&A[\mathfrak{b}_i/x_{\alpha_1}^{\alpha_1-i+1} x_{\alpha_2}^{\alpha_2-\alpha_1} \cdots x_{\alpha_k}^{s-\alpha_{k-1}+1}] \\
&\quad = A\left[\frac{x_i}{x_{\alpha_1}}, \ldots, \frac{x_{\alpha_1-1}}{x_{\alpha_1}}, \frac{x_{\alpha_2}}{x_{\alpha_1}}, \frac{x_{\alpha_1+1}}{x_{\alpha_2}}, \cdots, \frac{x_{\alpha_{k-1}-1}}{x_{\alpha_{k-1}}}, \frac{x_{\alpha_k}}{x_{\alpha_{k-1}}}, \frac{x_{\alpha_{k-1}+1}}{x_{\alpha_k}}, \cdots, \frac{x_d}{x_{\alpha_k}}\right]
\end{aligned}$$

with $i \leq \alpha_1 < \cdots < \alpha_{k-1} \leq s < \alpha_k \leq d$. Assume that $i > t$. Then it is easy to verify that

$$\begin{aligned}
A[\mathfrak{b}_{i-1}/x_\alpha^{s-i+3}] &= A[\mathfrak{b}_i/x_\alpha^{s-i+2}][x_{i-1}/x_\alpha], \\
A[\mathfrak{b}_{i-1}/x_{i-1}x_\alpha^{s-i+2}] &= A[\mathfrak{b}_i/x_\alpha^{s-i+2}][x_\alpha/x_{i-1}], \\
A[\mathfrak{b}_{i-1}/x_{\alpha_1}^{\alpha_1-i} \cdots x_{\alpha_k}^{s-\alpha_{k-1}+1}] &= A[\mathfrak{b}_i/x_{\alpha_1}^{\alpha_1-i+1} \cdots x_{\alpha_k}^{s-\alpha_{k-1}+1}][x_{i-1}/x_{\alpha_1}], \\
A[\mathfrak{b}_{i-1}/x_{i-1}x_{\alpha_1}^{\alpha_1-i+1} \cdots x_{\alpha_k}^{s-\alpha_{k-1}+1}] &= A[\mathfrak{b}_i/x_{\alpha_1}^{\alpha_1-i+1} \cdots x_{\alpha_k}^{s-\alpha_{k-1}+1}][x_{\alpha_1}/x_{i-1}], \\
\mathfrak{q}_i A[\mathfrak{b}_i/x_\alpha^{s-i+2}] &= x_\alpha A[\mathfrak{b}_i/x_\alpha^{s-i+2}]
\end{aligned}$$

and

$$\mathfrak{q}_i A[\mathfrak{b}_i/x_{\alpha_1}^{\alpha_1-i+1} \cdots x_{\alpha_k}^{s-\alpha_{k-1}+1}] = x_{\alpha_1} A[\mathfrak{b}_i/x_{\alpha_1}^{\alpha_1-i+1} \cdots x_{\alpha_k}^{s-\alpha_{k-1}+1}].$$

Therefore

Corollary 3.5. *The sheaf $\mathfrak{q}_{i-1}\mathcal{O}_{X_i}$ of ideals is locally generated by two elements and X_{i-1} is the blowing-up of X_i with respect to $\mathfrak{q}_{i-1}\mathcal{O}_{X_i}$ for all $t < i \leq s+1$.*

Now we prove Theorem 3.1 by induction on t. We may assume that $A/\mathfrak{m}$ is algebraically closed without loss of generality: see the proof of [6, Proposition 3.5].

If $t = s+1$, then $x_{s+1}, \ldots, x_d$ is a u.s.d-sequence on A. Let $R = A[\mathfrak{q}_{s+1}T]$ and $\mathfrak{M} = \mathfrak{m}R + R_+$. Then $H^i_{\mathfrak{M}}(R)$ is finitely graded for all $i \leq d-s$, that is, the homogeneous component $[H^i_{\mathfrak{M}}(R)]_n$ is zero for all but finitely many n. By using [3, Satz 1], we have depth $\mathcal{O}_{X_{s+1},p} \geq d-s$ for all closed point $p \in X_{s+1}$.

Next we assume that $t \leq s$ and let p be a closed point of X_t. Since the blowing-up $X_t \to \operatorname{Spec} A$ is a closed map, we have an expression:

$$\mathcal{O}_{X_t,p} = A\left[\frac{x_t}{x_{\alpha_1}}, \frac{x_{t+1}}{x_{\alpha_1}}, \cdots, \frac{x_d}{x_{\alpha_k}}\right]_{(\mathfrak{m}, x_t/x_{\alpha_1}-a_t, x_{t+1}/x_{\alpha_1}-a_{t+1}, \ldots)}$$

(or $\mathcal{O}_{X_t,p} = A[\mathfrak{b}_t/x_\alpha^{s-t+2}]_{(\mathfrak{m}, x_t/x_\alpha - a_t, x_{t+1}/x_\alpha - a_{t+1}, \ldots)}$) with $a_t, a_{t+1}, \ldots \in A$. Assume that $\alpha_1 > t$ and let l be a positive integer. Let

$$B = A\left[\frac{x_{t+1}}{x_{\alpha_1}}, \cdots, \frac{x_d}{x_{\alpha_k}}\right]_{(\mathfrak{m}, x_t/x_{\alpha_1}-a_t, x_{t+1}/x_{\alpha_1}-a_{t+1}, \ldots)},$$

$$B^{(l)} = A/x_t^l A\left[\frac{x_{t+1}}{x_{\alpha_1}}, \cdots, \frac{x_d}{x_{\alpha_k}}\right]_{(\mathfrak{m}, x_t/x_{\alpha_1}-a_t, x_{t+1}/x_{\alpha_1}-a_{t+1}, \ldots)}$$

and $\mathfrak{n}$ be the maximal ideal of B. Since $x_{t+1}, \ldots, x_d$ satisfies (#) and (%) as a subsystem of parameters for A and for $A/x_t^l A$, the induction hypothesis says that depth B, depth $B^{(l)} \geq d-t$.

We compute $H^i_{\mathfrak{q}_t}(B)$. Since $\mathfrak{q}_t B$ is generated by x_t and x_{α_1}, which are non-zero divisor on B, we have $H^q_{\mathfrak{q}_t}(B) = 0$ for $q \neq 1, 2$. Taking direct limit, local cohomology with respect to x_{α_1} and localization of a short exact sequence

$$(3.5.1) \qquad 0 \longrightarrow \bigoplus_{n>0} \frac{\mathfrak{b}_{t+1}^n : x_t^l}{\mathfrak{b}_{t+1}^n + 0 : x_t} \xrightarrow{x_t^l} \bigoplus_{n>0} \frac{\mathfrak{b}_{t+1}^n}{x_t^l \mathfrak{b}_{t+1}^n} \longrightarrow \bigoplus_{n>0} \frac{\mathfrak{b}_{t+1}^n + (x_t^l)}{(x_t^l)} \longrightarrow 0,$$

we obtain $H^1_{x_{\alpha_1}} H^1_{x_t}(B) = \varinjlim_{l,m} B^{(l)}/x_{\alpha_1}^m B^{(l)}$ because the left term of (3.5.1) is annihilated by x_{α_1}: see Proposition 3.3. The spectral sequence $E_2^{pq} = H^p_{x_t} H^q_{x_{\alpha_1}}(-) \Rightarrow H^n_{(x_t, x_{\alpha_1})}(-)$ induces a short exact sequence

$$0 \to H^1_{x_{\alpha_1}} H^{p-1}_{x_t}(-) \to H^p_{(x_t, x_{\alpha_1})}(-) \to H^0_{x_{\alpha_1}} H^p_{x_t}(-) \to 0.$$

Hence $H^2_{\mathfrak{q}_t}(B) = \varinjlim_{l,m} B^{(l)}/x_{\alpha_1}^m B^{(l)}$. Since x_{α_1} is a non-zero divisor on $B^{(l)}$,

$$H^p_{\mathfrak{n}} H^2_{\mathfrak{q}_t}(B) = 0 \quad \text{for all } p < d-t-1.$$

Furthermore, we get

$$H^1_{\mathfrak{q}_t}(A[\mathfrak{b}_t T]_+) = H^0_{x_{\alpha_1}} H^1_{x_t}(A[\mathfrak{b}_t T]_+) = \bigoplus_{n>0} \frac{\mathfrak{b}_{t+1}^n : x_t}{\mathfrak{b}_{t+1}^n + 0 : x_t},$$

from (3.5.1). In fact, x_{α_1} is a non-zero divisor on the right term of (3.5.1) because $(x_t^l) : x_{\alpha_1} \cap [(x_t^l) + \mathfrak{q}_{\alpha_1}] = (x_t^l)$. Therefore $\mathfrak{q}_t H^1_{\mathfrak{q}_t}(B) = 0$.

Consider the spectral sequence $E_2^{pq} = H^p_{\mathfrak{n}} H^q_{\mathfrak{q}_t}(-) \Rightarrow H^n_{\mathfrak{n}}(-)$. Since depth $B \geq d-t$, $E_2^{p1} = H^{p+1}_{\mathfrak{n}}(B) = 0$ for $p < d-t-1$. Thus

$$H^p_{\mathfrak{n}} H^q_{\mathfrak{q}_t}(B) = 0 \quad \text{if } q \neq 1, 2 \text{ or } p < d-t-1$$

and

$$\mathfrak{q}_t H^1_{\mathfrak{q}_t}(B) = 0.$$

By using this, we compute the depth of

$$\mathcal{O}_{X_t,p} = B[x_t/x_{\alpha_1}]_{(\mathfrak{n}, x_t/x_{\alpha_1} - a_t)} \cong \left(\frac{B[U]}{\bigcup_{l>0}(x_{\alpha_1}U - x_t) : x^l_{\alpha_1}} \right)_{(\mathfrak{n}, U - a_t)},$$

where U denotes an indeterminate. Taking local cohomology with respect to (x_t, x_{α_1}) of a short exact sequence

$$0 \longrightarrow B[U] \xrightarrow{x_{\alpha_1}U - x_t} B[U] \longrightarrow B[U]/(x_{\alpha_1}U - x_t) \longrightarrow 0,$$

we obtain an exact sequence

$$0 \longrightarrow H^1_{\mathfrak{q}_t}(B[U]) \longrightarrow H^1_{\mathfrak{q}_t}(B[U]/(x_{\alpha_1}U - x_t)) \longrightarrow H^2_{\mathfrak{q}_t}(B[U]) \longrightarrow H^2_{\mathfrak{q}_t}(B[U]) \longrightarrow 0.$$

By using an exact sequence

$$0 \longrightarrow H^1_{(U-a_t)} H^{p-1}_{\mathfrak{n}}(-) \longrightarrow H^p_{(U-a_t,\mathfrak{n})}(-) \longrightarrow H^0_{(U-a_t)} H^p_{\mathfrak{n}}(-) \longrightarrow 0,$$

we get $H^p_{(\mathfrak{n}, U-a_t)} H^q_{\mathfrak{q}_t}(B[U]) = 0$ if $q \neq 1, 2$ or $p < d-t$. Hence we obtain

$$H^p_{(\mathfrak{n}, U-a_t)} H^1_{\mathfrak{q}_t}(B[U]/(x_{\alpha_1}U - x_t)) = 0 \quad \text{for } p < d-t.$$

Taking local cohomology of a short exact sequence

$$0 \longrightarrow \frac{\bigcup_{l>0}(x_{\alpha_1}U - x_t) : x^l_{\alpha_1}}{(x_{\alpha_1}U - x_t)} \longrightarrow \frac{B[U]}{(x_{\alpha_1}U - x_t)} \longrightarrow B[x_t/x_{\alpha_1}] \longrightarrow 0,$$

we have

$$H^1_{\mathfrak{q}_t}(B[x_t/x_{\alpha_1}]) \cong H^1_{\mathfrak{q}_t}(B[U]/(x_{\alpha_1}U - x_t))$$

that is,

$$H^p_{(\mathfrak{n}, x_t/x_{\alpha_1} - a_t)} H^1_{\mathfrak{q}_t}(B[x_t/x_{\alpha_1}]) = 0 \quad \text{for } p < d-t.$$

Of course, $H^q_{\mathfrak{q}_t}(B[x_t/x_{\alpha_1}]) = 0$ if $q \neq 1$. The spectral sequence

$$E_2^{pq} = H^p_{(\mathfrak{n}, x_t/x_{\alpha_1} - a_t)} H^q_{\mathfrak{q}_t}(B[x_t/x_{\alpha_1}]) \Rightarrow H^n_{(\mathfrak{n}, x_t/x_{\alpha_1} - a_t)}(B[x_t/x_{\alpha_1}])$$

says that depth $\mathcal{O}_{X_t,p} \geq d-t+1$.

If $\alpha_1 = t$ or $\mathcal{O}_{X_t,p} = A[\mathfrak{b}_t/x_\alpha^{s-t+2}]_{(\mathfrak{m}, x_t/x_\alpha - a_t, \ldots)}$ then we can also show depth $\mathcal{O}_{X_t,p} \geq d-t+1$ in the same way as above. Thus Theorem 3.1 is proved.

4. The Proof of Theorem 1.2

In this section, we prove Theorem 1.2 in the same way as [1]. Let A be a Noetherian local ring possessing a dualizing complex and $s = \dim \text{non-CM}\, A$.

First we assume $\operatorname{Ass} A = \operatorname{Assh} A$. We work by induction on s. If $s < 0$, that is, A is Cohen-Macaulay, then the idealization $A \ltimes K_A$ of the canonical module K_A, which exists because A possesses a dualizing complex, is a Gorenstein ring [11] and A is its homomorphic image.

When $0 \leq s \leq 2$, let $x_1, \dots, x_d$ be a p-standard system of parameters of type s for A, $\mathfrak{q}_i = (x_i, \dots, x_d)$ and $\mathfrak{b}_i = \mathfrak{q}_i \cdots \mathfrak{q}_{s+1}$ for $i \leq s+1$. We consider $R = A[\mathfrak{b}_1^{d-1}T]$ and $\mathfrak{M} = \mathfrak{m} + R_+$. If $s = 0$, then $R_{\mathfrak{M}}$ is Cohen-Macaulay [7, Theorem 7.11] and A is its homomorphic image. Since $R_{\mathfrak{M}}$ also possesses a dualizing complex, A is a homomorphic image of a Gorenstein ring.

Assume that $s > 0$ and let $\mathfrak{P} \subset R$ be a prime ideal such that $\dim R/\mathfrak{P} \geq s$. We show that $R_{\mathfrak{P}}$ is Cohen-Macaulay, hence $\dim \text{non-CM}\, R_{\mathfrak{M}} < s$. Without loss of generalities, we may assume that $\mathfrak{P}$ is homogeneous. If $\mathfrak{P} \not\supset R_+$, then $R_{\mathfrak{P}}$ is Cohen-Macaulay by Theorem 1.1. If $\mathfrak{P} \supset R_+$, then we put $\mathfrak{P} = \mathfrak{p}R + R_+$ with $\mathfrak{p} \in \operatorname{Spec} A$. If $\mathfrak{p} \not\supset \mathfrak{q}_{s+1}$, then $R_{\mathfrak{p}} = A_{\mathfrak{p}}[T]$ is Cohen-Macaulay. If $\mathfrak{p} \supset \mathfrak{q}_{s+1}$, then $x_{s+1}, \dots, x_d$ is a system of parameters for $A_{\mathfrak{p}}$ which forms a u.s.d-sequence on $A_{\mathfrak{p}}$ because $\dim A/\mathfrak{p} = \dim R/\mathfrak{P} \geq s$. Hence $R_{\mathfrak{p}} = A_{\mathfrak{p}}[\mathfrak{q}_{s+1}^{d-1}A_{\mathfrak{p}}T]$ is Cohen-Macaulay. By induction hypothesis, we find that $R_{\mathfrak{M}}$ is a homomorphic image of a Gorenstein ring and A is also.

Next we consider the general case, we work by induction on $d = \dim A$. If $d = 0$, then there is nothing to prove. Assume that $d > 0$. Let $(0) = \mathfrak{r}_1 \cap \cdots \cap \mathfrak{r}_n$ be a primary decomposition of (0) in A. By renumbering $\mathfrak{r}_i$, we may assume that there is an integer $l \leq n$ such that $\dim A/\mathfrak{r}_i = d$ if and only if $i \leq l$. Let $\mathfrak{f} = \mathfrak{r}_1 \cap \cdots \cap \mathfrak{r}_l$ and $\mathfrak{f}' = \mathfrak{r}_{l+1} \cap \cdots \cap \mathfrak{r}_n$.

Let $\mathfrak{p}$ such that $\dim A/\mathfrak{p} \geq s$. Then $A_{\mathfrak{p}}$ is Cohen-Macaulay, hence equidimensional. Therefore $\mathfrak{p} \supset \mathfrak{f}$ if and only if $\mathfrak{p} \not\supset \mathfrak{f}'$. This implies that $\dim \text{non-CM}\, A/\mathfrak{f}$, $\dim \text{non-CM}\, A/\mathfrak{f}' \leq s$. By induction hypothesis and the case of $\operatorname{Ass} A = \operatorname{Assh} A$, there are Gorenstein local rings B and B' such that $A/\mathfrak{f}$ and $A/\mathfrak{f}'$ are their homomorphic image, respectively. We may assume that $\dim B = \dim B' = d$.

Consider A as a subring of $A/\mathfrak{f} \oplus A/\mathfrak{f}'$. Let C be the inverse image of A by $B \oplus B' \to A/\mathfrak{f} \oplus A/\mathfrak{f}'$. Then there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r] \arrow[d, "g"'] & B \oplus B' \arrow[r] \arrow[d, "f"'] & B \oplus B'/C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & A/\mathfrak{f} \oplus A/\mathfrak{f}' \arrow[r] & A/\mathfrak{f} + \mathfrak{f}' \arrow[r] & 0
\end{tikzcd}$$

with exact rows and epimorphisms f and g.

Since $A/\mathfrak{f} + \mathfrak{f}'$ is finitely generated over A, $B \oplus B'/C$ and $B \oplus B'$ are finitely generated over C. Therefore C is a Noetherian local ring by Eakin-Nagata theorem.

Since

$$\begin{array}{ccc} C & \longrightarrow & B' \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \oplus B'/C \end{array}$$

is a fiber product, B possesses a dualizing complex: see [5, Lemma 3 and 5] or [10, Corollary 3.7]. Furthermore, $\dim \text{non-CM}\, C \leq s$ and $\operatorname{Ass} C = \operatorname{Assh} C$ because $B \oplus B'$ is a Cohen-Macaulay C-module and $\dim A/\mathfrak{f} + \mathfrak{f}' \leq s$. Thus C is a homomorphic image of a Gorenstein local ring and A is also.

References

1. Yoichi Aoyama and Shiro Goto, *A conjecture of Sharp—the case of local rings with* $\dim \text{non-CM} \leq 1$ *or* $\dim \leq 5$, Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata, Kinokuniya, 1987, pp. 27–34.
2. Nguyen Tu Cuong, *On the dimension of the non-Cohen-Macaulay locus of local rings admitting dualizing complexes*, Math. Proc. Cambridge Philos. Soc. **109** (1991), 479–488.
3. Gerd Faltings, *Über die Annulatoren lokaler Kohohomogiegruppen*, Arch. Math. (Basel) **30** (1978), 473–476.
4. ______, *Über Macaualyfizierung*, Math. Ann. **238** (1978), 175–192.
5. ______, *Zur Existenz dualisierender Komplexe*, Math. Z. **162** (1978), 75–86.
6. Shiro Goto, *Blowing-up of Buchsbaum rings*, Prceedings, Durham symposium on Commutative Algebra, London Math. Soc. Lect. Notes, vol. 72, Cambridge Univ. Press, 1982, pp. 140–162.
7. Shiro Goto and Kikumichi Yamagishi, *The theory of unconditioned strong d-sequences and modules of finite local cohomology.*
8. Heisuke Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic 0*, Ann. of Math. **79** (1964), 109–326.
9. Takesi Kawasaki, *On Macaulayfication of certain quasi-projective schemes*, preprint, 1995.
10. Tetsushi Ogoma, *Existence of dualizing complexes*, J. Math. Kyoto Univ. **24** (1984), 27–48.
11. Idun Reiten, *The converse to a theorem of Sharp in Gorenstein modules*, Proc. Amer. Math. Soc. **32** (1972), 417–420.
12. Peter Schenzel, *Dualizing complexes and system of paramters*, J. Algebra **58** (1979), 495–501.
13. ______, *Cohomological annihilators*, Math. Proc. Cambridge Philos. Soc. **91** (1982), 345–350.
14. Rodney Y. Sharp, *Necessary conditios for the existence of dualizing complexes in commutative algebra*, Sém. Algèbre P. Dubreil 1977/78, Lecture Notes in Mathematics, vol. 740, Springer-Verlag, 1979, pp. 213–229.

Department of Mathematics, Tokyo Metropolitan University, Hachioji Minami-Ohsawa 1-1, Tokyo 192-03 Japan

E-mail address: kawasaki@math.metro-u.ac.jp